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Auslander–Reiten sequences over derivation polynomial rings

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Abstract

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Let F be a field with derivation $\xi \mapsto \xi'$ and $R = F[Y, ']$ the corresponding derivation polynomial ring. In this article we provide necessary and sufficient conditions for the existence of enough Auslander–Reiten sequences in the category $\mathfrak{m}_R$ of R-modules of finite length. Since in some respects R-modules can be treated in the same way like modules over a group algebra, only the existence of an Auslander–Reiten sequence in $\mathfrak{m}_R$ ending with the simple module R/YR has to be explored. Furthermore, we show that certain generalized power series fields F satisfy the characteristic conditions of the existence theorem: a typical example is the field $k((X))$ of formal Laurent series over a field k of characteristic 0.

Introduction

Let F be a field supplied with a derivation $\xi \mapsto \xi'$, $R = F[Y, ']$ the corresponding derivation polynomial ring and $\mathfrak{m}_R$ the category of all R-modules of finite length. The aim of this article is to provide criteria for the existence of enough Auslander–Reiten sequences in the category $\mathfrak{m}_R$. This problem, besides being of interest of its own, originates from the fact that $\mathfrak{m}_R$ plays an important role in the representation theory of the triangular matrix ring $S = \left(\begin{smallmatrix} F & {}_FN_F \\ 0 & F \end{smallmatrix}\right)$, the bimodule ${}_FN_F$ being given by ${}_FN = {}_FF \times {}_FF$ and by $(\alpha, \beta)\xi = (\alpha\xi + \beta\xi', \beta\xi)$ for $(\alpha, \beta) \in N$ and $\xi \in F$. More precisely, Ringel has proved that the category of all regular S-modules is a product of two subcategories $\mathfrak{u} \times \mathfrak{m}$, where $\mathfrak{u}$ is a uniserial category of global dimension 1 with exactly one simple object and $\mathfrak{m}$ is equivalent to $\mathfrak{m}_R$ [7, 7.4]. In our paper [9] we have shown that the category of all finitely generated S-modules possesses enough Auslander–Reiten sequences provided F is the field of formal Laurent series in one variable over a field of characteristic 0; the bulk of the proof was to show this for the modules in $\mathfrak{m}$. The most remarkable feature of this result is the fact that the Auslander–Reiten sequences of the modules in $\mathfrak{m}$ are not Auslander–Reiten sequences in the whole module category. These results as well as their rather intricate and technical proofs are considerably clarified and generalized in the present paper. This is partly achieved by the observation that the category of R-modules is very similar to a module category over a group algebra. Thus certain results by Auslander and Carlson concerning Auslander–Reiten sequences over group algebras [2] can be transferred to R-modules.

We give a short summary of the paper. Since our results are looking smoothest over fields of characteristic 0 satisfying $F' \neq F$, we momentarily restrict ourselves to this case. First it is demonstrated that there exists an Auslander–Reiten sequence $0 \to A \to B \to M \to 0$ in $\mathfrak{m}_R$ for each $M \in \mathfrak{m}_R$ iff there exists an Auslander–Reiten sequence $0 \to A' \to B' \to F \to 0$ in $\mathfrak{m}_R$; here F_R denotes the simple module R/YR. Moreover, in this case $A \cong M$ and $A' \cong F_R$, and the sequence $0 \to A \to B \to M \to 0$ is a tensor product of $0 \to A' \to B' \to F \to 0$ by M. Consequently it has only to be explored when there exists an Auslander–Reiten sequence in $\mathfrak{m}_R$ ending with F_R. We show that this holds if and only if F/F' is one-dimensional over the field of constants and each injective differential operator $F \to F$, $\xi \mapsto r_* \xi$, induced by some $r \in R$, is surjective. Finally we specify a class of derivation fields which share each of these properties. The fields in question are the fields of power series $k((\Gamma))$, where k is a field and Γ a linearly ordered group contained in the additive group of k. Perhaps the best known example is the field of Laurent series $k((X))$ in one variable over a field k of characteristic 0 (in this case $\Gamma = \mathbb{Z}$).

At last we recall a definition and fix a notation. Let $\mathfrak{A}$ be a full subcategory of the category of all modules over some ring. Then a non split exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathfrak{A}$ is called an Auslander–Reiten sequence (AR-sequence for short) in $\mathfrak{A}$, if the following conditions are met:

(i) Each morphism $f' : A \to A'$ in $\mathfrak{A}$ which is not a split monomorphism factors through f.

(ii) Each morphism $g' : C' \to C$ in $\mathfrak{A}$ which is not a split epimorphism factors through g.

The Jacobson radical of a ring S will be denoted by $J(S)$.

1. General facts on Auslander–Reiten sequences over $F[Y, ']$

We start with a review of facts on derivation polynomial rings. Let F denote a field with a derivation $F \to F$, $\xi \mapsto \xi'$. The corresponding derivation polynomial ring $R = F[Y, ']$ is defined as the left vector space $\bigoplus_{i \in \mathbb{N}} FY^i$, the multiplication being fixed by $Y\xi = \xi Y + \xi'$. Then $(Y^i)_{i \in \mathbb{N}}$ is a right F-basis of R as well. The degree of an element $r = \sum_{i=0}^{n} \xi_i Y^i \in R$ with $\xi_n \neq 0$ is $\deg r = n$; furthermore, $\deg 0 = -\infty$. It follows that there holds an Euclidean algorithm in R. For all

$r,p \in R$, $p \neq 0$, there are unique elements $q,s \in R$ such that $r = pq + s$ and $\deg s < \deg p$, and unique $q_1,s_1 \in R$ satisfying $r = q_1 p + s_1$, $\deg s_1 < \deg p$. Hence R is a principal ideal ring on either side. As a consequence, each finitely generated right R-module is isomorphic to $R^n \times \prod_{i=1}^{m} R/r_iR$ for suitable numbers $m,n \in \mathbb{N}$ and elements $r_1, \ldots, r_m \in R\backslash\{0\}$. This implies that a monomorphism $f : A_R \to B_R$ is pure if and only if $f(A)r = f(A) \cap Br$ for all $r \in R$. R carries an involution $\iota : R \to R$, where

$$\iota(r) = \tilde{r} = \sum_{i=0}^{n} (-1)^i Y^i \alpha_i \quad \text{for } r = \sum_{i=0}^{n} \alpha_i Y^i ,$$

respectively

$$\iota(s) = \tilde{s} = \sum_{j=0}^{m} (-1)^j \beta_j Y^j \quad \text{for } s = \sum_{j=0}^{m} Y^j \beta_j .$$

By means of ι each R-module M can be made an R-module $\tilde{M}$ on the opposite side. For instance, $\tilde{M} \cong R/R\tilde{r}$, if $M = R/rR$ with $r \in R$. In the following the simple module R/YR which we identify with F will play an outstanding role. As for the operation of R on F, we shall write $\xi_* r$ for $\xi \in F$ and $r \in R$; it is completely fixed by $\xi_* Y = -\xi'$. Similarly we write $r_* \xi$ for the operation of R on $_R F = \tilde{F}$; it is determined by $Y_* \xi = \xi'$. Thus each $r \in R$ yields differential operators $_* r : F \to F$, $\xi \mapsto \xi_* r$, and $r_* : F \to F$, $\xi \mapsto r_* \xi$, which are obviously k-linear, $k = \{\xi \in F \mid \xi' = 0\}$ denoting the field of constants of our derivation. Note that $r_* \xi = \xi_* \tilde{r}$. By use of the Euclidean algorithm it is easily checked that $\xi_* r = 0$ if and only if $\xi r \in YR$.

Next we compile certain operations with R-modules needed in the sequel (see [4]). Similar operations are also known for instance for modules over group algebras. We denote the category of all right and left R-modules by Mod-R and R-Mod, and the full subcategories of all modules of finite length by $\mathfrak{m}_R$ and $_R\mathfrak{m}$, respectively; note that an R-module is of finite length iff it is finite-dimensional over F. For $M,N \in$ Mod-R the vector space $\mathrm{Hom}_F(M, N)$ is an R-module by $(fY)(x) = -f(xY) + f(x)Y$ for $f \in \mathrm{Hom}_F(M, N)$ and $x \in M$; we denote it by $\mathrm{Hom}_F(M, N)_R$ or simply by $\mathrm{Hom}_F(M, N)$. The case $N = F_R$ is of particular importance; we put $DM = \mathrm{Hom}_F(M, F)_R$, the module structure being given by $(fY)(x) = -f(xY) - f(x)'$. Because the evaluation map $M \to D^2M$ is an isomorphism for all $M \in \mathfrak{m}_R$, D induces a duality $D : \mathfrak{m}_R \to \mathfrak{m}_R$.

Lemma 1. *There is an isomorphism $D(R/aR) \cong R/\tilde{a}R$ for all $0 \neq a \in R$. In particular, $D(F_R) \cong F_R$.*

Proof. Letting $a = \sum_{i=0}^{n} Y^i \alpha_i$ with $\alpha_n = 1$, the residue classes $\bar{1}, \bar{Y}, \ldots, \bar{Y}^{n-1}$ form an F-basis of $M = R/aR$. The elements $f_0, f_1, \ldots, f_{n-1} \in DM$ of the corresponding dual basis satisfy the equations $f_i Y = -f_{i-1} + f_{n-1}\alpha_i$, $0 \leq i \leq n-1$ (with

$f_{-1}=0$). Inserting yields $f_{n-1}\tilde{a}=0$; thus the map $\varphi: R \to DM$, $r \mapsto f_{n-1}r$, is an epimorphism whose kernel contains $\tilde{a}R$. Since the dimensions of $R/\tilde{a}R$ and DM coincide, φ induces an isomorphism $R/\tilde{a}R \to DM$. $\square$

In this context we recall the definition of the transposed of a cyclic module in $\mathfrak{m}_R$. Let $M = R/aR$ with $0 \neq a \in R$, then the sequence $0 \to R_R \xrightarrow{u} R_R \xrightarrow{\text{can}} M_R \to 0$ is exact, where u is left multiplication by a. Application of the functor $\operatorname{Hom}_R(-, R_R)$ and identification yields the exact sequence $0 \to {}_RR \xrightarrow{u^*} {}_RR \xrightarrow{\text{can}} \operatorname{Tr} M \to 0$ with $u^*(r) = ra$, hence $\operatorname{Tr} M = R/Ra$. Note that

$$\widetilde{D(R/aR)} \cong \widetilde{R/\tilde{a}R} \cong R/Ra \cong \operatorname{Tr}(R/aR).$$

The tensor product $(M \otimes_F N)_R$ of two modules $M, N \in \text{Mod-}R$ is the vector space $M \otimes_F N$ supplied with the scalar multiplication $(x \otimes y)Y = (xY) \otimes y + x \otimes (yY)$ for $x \in M$, $y \in N$. Obviously it is commutative and associative, and $(M \otimes_F F)_R \cong M_R \cong (F \otimes_F M)_R$ canonically. In the next lemma we shall collect further canonical morphisms between diverse Hom- and $\otimes$-functors (cf. [2]).

Lemma 2. *Let $M, N, P \in \text{Mod-}R$.*

(1) (a) $\Phi_1 : (M \otimes_F N)_R \to \operatorname{Hom}_F(DM, N)_R$, $\Phi_1(x \otimes y)(f) = y \cdot f(x)$, *is a linear map.*

(b) *If $M \in \mathfrak{m}_R$, then Φ_1 and $\Phi_1' : (DM \otimes_F N)_R \to \operatorname{Hom}_F(M, N)_R$, $\Phi_1'(f \otimes y)(x) = y \cdot f(x)$, are isomorphisms.*

(2) $\Phi_2 : \operatorname{Hom}_R((M \otimes_F N)_R, P) \to \operatorname{Hom}_R(M, \operatorname{Hom}_F(N, P)_R)$, $\Phi_2(h)(x)(y) = h(x \otimes y)$, *is an isomorphism.*

(3) *If $P \in \mathfrak{m}_R$, then there exists an isomorphism $\Phi_3 : \operatorname{Hom}_R(M, (N \otimes_F P)_R) \to \operatorname{Hom}_R(\operatorname{Hom}_F(P, M)_R, N)$, which is defined as follows. Let $(p_i, \psi_i)_{1 \le i \le n}$ be a dual basis of P_F, let $f \in \operatorname{Hom}_R(M, (N \otimes_F P)_R)$, $h \in \operatorname{Hom}_F(P, M)$ and $f(h(p_i)) = \sum_{j=1}^n y_{ij} \otimes p_j$ with $y_{ij} \in N$, then $\Phi_3(f)(h) = \sum_{i=1}^n y_{ii}$.*

(4) *For $M \in \mathfrak{m}_R$ there is an isomorphism $\Phi_4 : \operatorname{Hom}_F(M, N \otimes_F P)_R \to (\operatorname{Hom}_F(M, N) \otimes_F P)_R$ which is given as follows. Let $(x_i, \varphi_i)_{1 \le i \le n}$ denote a dual basis of M_F, $(p_j)_{j \in J}$ a basis of P_F, let $f \in \operatorname{Hom}_F(M, N \otimes_F P)$ and $f(x_i) = \sum_j n_{ij} \otimes p_j$ with $n_{ij} \in N$. Then the maps $t_j \in \operatorname{Hom}_F(M, N)$ defined by $t_j(x) = \sum_i n_{ij} \cdot \varphi_i(x)$ are zero for almost all $j \in J$; put $\Phi_4(f) = \sum_j t_j \otimes p_j$.*

Proof. We only note that the inverse of Φ_1 is given by $\operatorname{Hom}_F(DM, N) \to M \otimes_F N$, $h \mapsto \sum_{i=1}^n x_i \otimes h(\varphi_i)$, where $(x_i, \varphi_i)_{1 \le i \le n}$ is a dual basis of M_F and that (3) and (4) are consequences of (1) and (2). $\square$

Choosing $M = P \in \mathfrak{m}_R$ and $N = F_R$ in (3), we obtain the isomorphism $\Phi_3 : \operatorname{End}_R(M) \to \operatorname{Hom}_R(\operatorname{End}_F(M)_R, F_R)$, $\Phi_3(f)(h) = \operatorname{tr}(fh)$, where tr denotes the usual trace of a linear map. In particular, $\operatorname{tr} : \operatorname{End}_F(M)_R \to F_R$ is R-linear.

Lemma 3 (cf. [2, Proposition 3.1]). *If $M \in \mathfrak{m}_R$ is indecomposable, then $\Phi_3(f)$ is not a split epimorphism for all $f \in J\,\mathrm{End}_R(M)$.*

Proof. If $\Phi_3(f)$ is a split epimorphism, then there is a linear map $\varphi : F_R \to \mathrm{End}_F(M)_R$ with $\Phi_3(f)\varphi = 1_F$. The equation $\varphi(1)Y = \varphi(1_* Y) = 0$ shows that $h = \varphi(1)$ is an R-linear map satisfying $\mathrm{tr}(fh) = \Phi_3(f)(h) = 1$. Hence fh cannot be nilpotent, i.e. $f \not\in J\,\mathrm{End}_R(M)$. □

Lemma 4. *Let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence in* Mod-R. *Then the tensored sequence*

$$0 \to (M' \otimes_F N)_R \to (M \otimes_F N)_R \to (M'' \otimes_F N)_R \to 0$$

is also pure for all $N \in \mathfrak{m}_R$.

Proof. Letting $X \in \mathfrak{m}_R$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(X, (M' \otimes_F N)_R) & \longrightarrow & \mathrm{Hom}_R(X, (M \otimes_F N)_R) & \longrightarrow & \mathrm{Hom}_R(X, (M'' \otimes_F N)_R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi_3} & & \downarrow{\scriptstyle \Phi_3} & & \downarrow{\scriptstyle \Phi_3} & & \\ 0 & \to & \mathrm{Hom}_R(\mathrm{Hom}_F(N, X)_R, M') & \to & \mathrm{Hom}_R(\mathrm{Hom}_F(N, X)_R, M) & \to & \mathrm{Hom}_R(\mathrm{Hom}_F(N, X)_R, M'') & \to & 0 \end{array}$$

where the vertical maps Φ_3 are isomorphisms by Lemma 2. Since $\mathrm{Hom}_F(N, X)_R$ lies in $\mathfrak{m}_R$, the lower row is exact, hence the upper one as well. □

Lemma 5 (cf. [2, Proposition 2.3]). *Given a non split exact sequence $\mathscr{E} : 0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ in* Mod-R, *the tensored sequence*

$$\mathscr{E} \otimes_F N : 0 \to (M' \otimes_F N)_R \to (M \otimes_F N)_R \to (M'' \otimes_F N)_R \to 0$$

does not split as well for each $N \in \mathfrak{m}_R$ such that $(\mathrm{char}\, K, \dim_F N) = 1$.

Proof. Let $N \in \mathfrak{m}_R$ be a module with $(\mathrm{char}\, F, \dim_F N) = 1$. According to Lemma 2 the map $\alpha_X : X_R \to \mathrm{Hom}_F(N, X \otimes_F N)_R$, $\alpha(x)(y) = x \otimes y$, is R-linear for all $X \in \mathrm{Mod}\text{-}R$, and the composition

$$X_R \xrightarrow{\alpha_X} \mathrm{Hom}_F(N, X \otimes_F N)_R \xrightarrow{\Phi_4} (\mathrm{Hom}_F(N, N) \otimes_F X)_R \xrightarrow{\mathrm{tr} \otimes 1} X_R$$

is nothing but multiplication by $\dim_F N$, hence α_X is a split monomorphism. If we assume that $\mathscr{E} \otimes N$ splits, then the induced sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_F(N, M' \otimes_F N)_R &\xrightarrow{\tilde{u}} \mathrm{Hom}_F(N, M \otimes_F N)_R \\ &\xrightarrow{\tilde{v}} \mathrm{Hom}_F(N, M'' \otimes N) \to 0 \end{aligned}$$

splits as well, hence $\tilde{u} \circ \alpha_{M'} = \alpha_M \circ u$ and u are split monomorphisms, a contradiction. □

For handling Auslander–Reiten sequences we shall need a second type of dualization. Given $M \in$ Mod-R we put $M^+ = \mathrm{Hom}_k(\tilde{M}, k)$; recall that k denotes the field of constants $\{\xi \in F \mid \xi' = 0\}$. The module structure of M^+ is induced by the left module structure of $\tilde{M}$, i.e. $(\varphi Y)(x) = \varphi(\tilde{Y}x) = -\varphi(xY)$ for $\varphi \in \mathrm{Hom}_k(M, k)$ and $x \in M$. For instance, the structure of $(F_R)^+$ is defined by $(\varphi r)(\xi) = \varphi(r_* \xi)$ for $\varphi \in F^+$, $r \in R$ and $\xi \in F$.

Lemma 6. *Let $M, N \in$ Mod-R.*

(1) *The map $\Phi_5 : (M \otimes_F N)^+ \to \mathrm{Hom}_F(N, M^+)_R$, $\Phi_5(\varphi)(x)(y) = \varphi(x \otimes y)$, is an R-isomorphism.*

(2) *In case $M \in \mathfrak{m}_R$, the map $\Phi_6 : (M \otimes_F N^+)_R \to \mathrm{Hom}_F(M, N)^+$, given by $\Phi_6(x \otimes \varphi)(\psi) = \varphi(\psi(x))$, is an isomorphism.*

Proof. (1) This is trivial.

(2) Φ_6 is composed of isomorphisms of type Φ_1' (Lemma 2) and Φ_5, hence an isomorphism, too. □

Now we turn to Auslander–Reiten sequences in the category $\mathfrak{m}_R$. Our starting point is a general existence theorem by Auslander which we formulate only for our special ring R.

Theorem 7 (Auslander [1, Theorem 3.9]). *Let $M \in \mathfrak{m}_R$ be indecomposable, $T = \mathrm{End}_R(M)$ and U_T an injective hull of the simple module $(T/J(T))_T$. Then there exists an Auslander–Reiten sequence $0 \to \mathrm{Hom}_T((\mathrm{Tr}\, M)_T, U_T) \to N \to M \to 0$ in* Mod-R. □

The first term $\mathrm{Hom}_T((\mathrm{Tr}\, M)_T, U_T)$ which, by the way, need not be finitely generated, admits a simpler description. Thanks to the following lemma the dimension of T over k is finite. Hence, as is well known, the injective hull of $T/J(T)$ equals $U_T = \mathrm{Hom}_k(T, k)$ and we obtain $\mathrm{Hom}_T(\mathrm{Tr}\, M, U_T) \cong \mathrm{Hom}_k(\mathrm{Tr}\, M, k) \cong (DM)^+$ (recall that $\widetilde{DM} \cong \mathrm{Tr}\, M$). In particular, $\mathrm{Hom}_T(\mathrm{Tr}(F_R), U_T) \cong F^+$; we shall denote by $\mathcal{F} : 0 \to F^+ \xrightarrow{f} Q \xrightarrow{g} F \to 0$ the corresponding Auslander–Reiten sequence.

The following lemma is known for special fields F (see for instance [6, Theorem 2.3] and [9, Theorem 15]). As we do not know whether it is known in full generality we include a proof.

Lemma 8. *The vector space $\mathrm{Hom}_R(M, N)$ is finite-dimensional over k for all $M, N \in \mathfrak{m}_R$.*

Proof. First we note that the kernel of the differential operator r_* is finite-dimensional over k for all $0 \neq r \in R$; more precisely, $\dim_k \operatorname{Ker} r_* \leq \deg r$ [5, p. 21]. Now denoting by $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ an F-basis of M and N respectively, the structure of M and N is fixed by a matrix $A \in F^{m \times m}$ and $B \in F^{n \times n}$ such that $(x_1, \ldots, x_m)Y = (x_1, \ldots, x_m)A$ and $(y_1, \ldots, y_n)Y = (y_1, \ldots, y_n)B$, respectively. Similarly, each $h \in \operatorname{Hom}_R(M, N)$ is represented by a matrix $H \in F^{n \times m}$ satisfying the equations $(h(x_1), \ldots, h(x_m)) = (y_1, \ldots, y_n)H$ and $h(x_iY) = h(x_i)Y$, $1 \leq i \leq m$, the latter being equivalent to $H' = BH - HA$. Interpreting H as an mn column over F, this equation can also be written in the form $H' = CH$ with a square matrix $C \in F^{mn \times mn}$. By derivating successively we obtain the equations $H^{(i)} = C_iH$, $i \geq 1$, where $C_1 = C$ and $C_{i+1} = C_i' + C_iC$ for $i \geq 1$. As $C_1, \ldots, C_{mn+1}$ are linearly dependent over F, there are $\alpha_1, \ldots, \alpha_{mn+1} \in F$, not all zero, with $\sum_{i=1}^{mn+1} \alpha_i C_i = 0$, hence

$$\sum_{i=1}^{mn+1} \alpha_i H^{(i)} = \left(\sum_{i=1}^{mn+1} \alpha_i C_i \right) H = 0 .$$

Putting $r = \sum_{i=1}^{mn+1} \alpha_i Y^i \in R$, this means $H \subset (\operatorname{Ker} r_*)^{mn}$, thus our first remark implies

$$\dim_k \operatorname{Hom}_R(M, N) \leq mn \cdot \dim_k \operatorname{Ker} r_* \leq mn(mn + 1) . \qquad \square$$

Next we cite a criterion for the existence of Auslander–Reiten sequences in the subcategory $\mathfrak{m}_R$, which subsequently is needed for the proof of two general theorems.

Theorem 9 ([8, Proposition 3] and [9, Theorem 2]). *For an indecomposable module $M \in \mathfrak{m}_R$, the following statements are equivalent*:

(1) *There exists an Auslander–Reiten sequence $0 \to A \to B \to M \to 0$ in $\mathfrak{m}_R$.*

(2) *There exists an indecomposable module $A \in \mathfrak{m}_R$ and a pure monomorphism $A \to (DM)^+$.*

Proof. In the cited papers this equivalence has been shown for artinian rings, but in the present context these proofs work as well. $\square$

We note that an Auslander–Reiten sequence $0 \to A \to B \to M \to 0$ in $\mathfrak{m}_R$ is not an Auslander–Reiten sequence in Mod-R if $\dim_k F$ is infinite. In fact, in this case the inequality of cardinals $\dim_k A = \dim_k F < \dim_k F^+ = \dim_k (DM)^+$ implies that the pure monomorphism $A \to (DM)^+$ cannot be an isomorphism.

Theorem 10. *Let $M \in \mathfrak{m}_R$ be indecomposable.*

(1) *The tensored sequence*

$$\mathcal{F} \otimes M : 0 \to F^+ \otimes_F M \xrightarrow{f \otimes 1} Q \otimes_F M \xrightarrow{g \otimes 1} M \to 0$$

is an Auslander–Reiten sequence in Mod-R *provided it does not split.*

(2) *Let* $\mathscr{E}: 0 \to A \to B \to F_R \to 0$ *denote an Auslander–Reiten sequence in* $\mathfrak{m}_R$. *If the sequence* $\mathscr{E} \otimes M$ *does not split, then it is an Auslander–Reiten sequence in* $\mathfrak{m}_R$ *as well.*

Proof. (1) (Cf. [2, Theorem 3.6].) Since $(F^+ \otimes_F M)_R \cong \mathrm{Hom}_F(M, F)^+ \cong (DM)^+$ by Lemma 6, it is sufficient to show that $g \otimes 1 : Q \otimes_F M \to M$ is right almost split, equivalently, that each $h \in J\,\mathrm{End}_R(M)$ factors through $g \otimes 1$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M, (Q \otimes_F M)_R) & \xrightarrow{\mathrm{Hom}(1, g\otimes 1)} & \mathrm{End}_R(M) \\ \Phi_3 \downarrow & & \downarrow \Phi_3 \\ \mathrm{Hom}_R(\mathrm{End}_F(M)_R, Q_R) & \xrightarrow{\mathrm{Hom}(1,g)} & \mathrm{Hom}_R(\mathrm{End}_F(M)_R, F_R) \end{array}$$

induced by isomorphisms of type Φ_3 in Lemma 2. As $\Phi_3(h)$ is not a split epimorphism for $h \in J\,\mathrm{End}_R(M)$ (Lemma 3), it factors through g, hence h factors through $g \otimes 1$.

(2) In this proof, our principal tool will be Theorem 9. Supposing that $0 \to A \to B \to F_R \to 0$ is an AR-sequence in $\mathfrak{m}_R$, there exists a pure monomorphism $\alpha : A \to F^+$. Then $\alpha \otimes 1 : (A \otimes_F M)_R \to (F^+ \otimes_F M)_R \cong (DM)^+$ is also pure by Lemma 4 and because $(DM)^+$ is indecomposable, $(A \otimes_F M)_R$ is indecomposable as well. Thus $\mathscr{E} \otimes M$ is an AR-sequence in $\mathfrak{m}_R$. □

Theorem 11. *If there exists an Auslander–Reiten sequence* $0 \to A \to B \to F_R \to 0$ *in* $\mathfrak{m}_R$, *then the module A has F-dimension one.*

Proof. As in the preceding proof there are pure monomorphisms $\alpha : A \to F^+$ and $\alpha \otimes 1 : (A \otimes_F DA)_R \to F^+ \otimes_F DA \cong A^+$. On the other hand, $D : \mathfrak{m}_R \to \mathfrak{m}_R$ being a duality, there is an AR-sequence $0 \to DF \cong F \to DB \to DA \to 0$ in $\mathfrak{m}_R$, hence a pure monomorphism $F_R \to (D^2A)^+ \cong A^+$. Using again Theorem 9 we may conclude $(A \otimes_F DA)_R \cong F_R$, hence $\dim_F A = 1$. □

2. Conditions for the existence of an Auslander–Reiten sequence $0 \to A \to B \to F \to 0$ in $\mathfrak{m}_R$

We have shown that if $0 \to A \to B \to F \to 0$ is an Auslander–Reiten sequence in $\mathfrak{m}_R$, then A has to be one-dimensional over F. Thus we may assume that $A = R/(Y - \alpha)R$ for some $\alpha \in F$. Next we give an explicit description of B.

Lemma 12. *Each non split exact sequence* $0 \to R/(Y-\alpha)R \xrightarrow{f} B \xrightarrow{g} F_R \to 0$ *is equivalent to an exact sequence*

$$0 \to R/(Y-\alpha)R \xrightarrow{u} R/Y\tfrac{1}{\lambda}(Y-\alpha)R \xrightarrow{v} F \to 0$$

where $0 \neq \lambda \in F$, $u(\bar{r}) = \overline{Y\frac{1}{\lambda}r}$, *and* v *is the canonical map. The latter sequence does not split if and only if* $\lambda \not\in (Y - \alpha)_* F$.

Proof. Let $x = f(\bar{1})$ and $y \in B$ with $g(y) = 1$. Since $g(yY) = 1_* Y = 0$, there is a unique $\lambda \in F$ with $yY = x\lambda$; as B is not semisimple, $\lambda \neq 0$. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & R/(Y-\alpha)R \arrow[r, "u"] \arrow[d, equal] & R/Y\frac{1}{\lambda}(Y-\alpha)R \arrow[r, "v"] \arrow[d, "s"] & F \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R/(Y-\alpha)R \arrow[r, "f"] & B \arrow[r, "g"] & F \arrow[r] & 0 \end{tikzcd}$$

commutes with $s(\bar{r}) = yr$, thus s has to be an isomorphism. It is easily seen that u splits iff there exists some $0 \neq \rho \in F$ with $1 - \rho Y\frac{1}{\lambda} \in (Y - \alpha)R$, i.e. if and only if $\lambda \in (Y - \alpha)_* F$. $\square$

All over this section let

$$\mathscr{E} : 0 \to R/(Y-\alpha)R \xrightarrow{u} R/Y\tfrac{1}{\lambda}(Y-\alpha)R \xrightarrow{v} F \to 0$$

denote the sequence in Lemma 12 and $U = (Y - \alpha)_* F$.

Theorem 13. *The following statements are equivalent*:

(1) $\mathscr{E}$ *is an Auslander–Reiten sequence in* $\mathfrak{m}_R$.

(2) *For each* $0 \neq a \in R$ *such that* $\xi a - 1 \not\in (Y - \alpha)R$ *for all* $\xi \in F$, *the corresponding differential operator* a_* *has a root in* $\lambda + U$.

If (1) *and* (2) *hold, then* $F = U \oplus k\lambda$.

Proof. (1)$\Rightarrow$(2) Suppose that $0 \neq a \in R$ satisfies $\xi a - 1 \not\in (Y - \alpha)R$ for all $\xi \in F$. Then $h : R/(Y - \alpha)R \to R/a(Y - \alpha)R$, $h(\bar{r}) = \overline{ar}$, is a non split monomorphism. Hence, $\mathscr{E}$ being an AR-sequence in $\mathfrak{m}_R$, there is a morphism $h' : R/Y\frac{1}{\lambda}(Y - \alpha)R \to R/a(Y - \alpha)R$ with $h'u = h$. Choosing a representative $c \in R$ of $h'(\bar{1})$, the latter equation means that $a - cY\frac{1}{\lambda} \in a(Y - \alpha)R$, equivalently $a(1 + (Y - \alpha)t) = cY\frac{1}{\lambda}$ for some $t \in R$. Hence $\lambda + (Y - \alpha)t_*\lambda$ is a root of a_* in $\lambda + U$.

(2)$\Rightarrow$(1) Now let $h : R/(Y - \alpha)R \to M$ be a monomorphism into an indecomposable module $M \in \mathfrak{m}_R$ which does not split. We have to prove that h factors through u. First we show that we can assume that $M = R/a(Y - \alpha)R$ for some $0 \neq a \in R$ and $h(\bar{r}) = \overline{ar}$ for all $\bar{r} \in R/(Y - \alpha)R$. Obviously we can assume that $M = R/bR$ for some $0 \neq b \in R$. Then the unique representative $a \in R$ of $h(\bar{1})$ with $\deg a < \deg b$ satisfies $a(Y - \alpha) \in bR$, Hence $a(Y - \alpha) = b\tau$ for some $0 \neq \tau \in F$, and we have $bR = a(Y - \alpha)R$. As h is not split, $\xi a - 1 \not\in (Y - \alpha)R$ for all $\xi \in F$, thus a_* has a root $\lambda + (Y - \alpha)_*\sigma \in \lambda + U$, i.e. there is a $t \in R$ with $a(\lambda + (Y - \alpha)\sigma) = tY$. Since $tY\frac{1}{\lambda}(Y - \alpha) = a((Y - \alpha) + (Y - \alpha)\frac{\sigma}{\lambda}(Y - \alpha)) \in a(Y - \alpha)R$, we can define a morphism $h' : R/Y\frac{1}{\lambda}(Y - \alpha)R \to M$ by $h'(\bar{r}) = \overline{tr}$, and obviously $h'u = h$.

To prove the last assertion, let $\mu \in F\backslash U$ and $a = Y\frac{1}{\mu} \in R$. Because an equation $\xi a - 1 = (Y-\alpha)t$ for $\xi \in F$ and $t \in R$ would lead to $\mu = -(Y-\alpha)t_*\mu \in U$, we have $\xi a - 1 \notin (Y-\alpha)R$ for all $\xi \in F$. Hence a_* has a root $\nu = \lambda + u \in \lambda + U$, which means that $(\frac{\nu}{\mu})' = 0$, i.e. $\kappa = \frac{\nu}{\mu}$ is an element of k. Since $\lambda \notin U$, we have $\kappa \neq 0$, hence $\mu = \frac{\nu}{\kappa} = \frac{\lambda}{\kappa} + \frac{u}{\kappa} \in k\lambda \oplus U$. $\square$

Lemma 14. *Let $F = U \oplus k\lambda$ with $\lambda \neq 0$ and $\pi : F \to k\lambda \cong k$ the corresponding projection. The operator a_*, $a \in R$, has no root in $\lambda + U$ if and only if there is $\varphi \in F^+$ with $\varphi a = \pi$.*

Proof. We have the following equivalences:

$$\begin{aligned} &a_* \text{ has no root in } \lambda + U \\ &\Leftrightarrow a_*\lambda \notin a_*U \\ &\Leftrightarrow \text{there exists } \psi \in F^+ \text{ with } \psi(a_*U) = 0 \text{ and } \psi(a_*\lambda) \neq 0 \\ &\Leftrightarrow \text{there exists } \psi \in F^+ \text{ and } 0 \neq \kappa \in k \text{ such that } \psi a = \pi\kappa \\ &\Leftrightarrow \text{there exists } \varphi \in F^+ \text{ with } \varphi a = \pi\,. \qquad \square \end{aligned}$$

Theorem 15. *Let $F = U \oplus k\lambda$ with $\lambda \neq 0$ and $\pi : F \to k$ the projection. Then the following statements are equivalent:*

(1) *$\mathscr{E}$ is an Auslander–Reiten sequence in $\mathfrak{m}_R$.*

(2) *If $\varphi \in F^+$, $a \in R$ and $\beta \in F$ satisfy the equation $\varphi a = \pi\beta$, then there is $\gamma \in F$ with $\gamma a - \beta \in (Y-\alpha)R$.*

(3) *The monomorphism $j : R/(Y-\alpha)R \to F^+$, $j(\bar{r}) = \pi r$, is pure.*

(4) *If a_* is injective for $a \in R$, then ${}_*a(Y+\alpha)$ is surjective.*

The 'internal' description (4) is the most advantageous one among these properties, since it can be tested in concrete cases (see Section 3). Note that the equivalence (1)$\Leftrightarrow$(3) is a special case of Theorem 9, for which we here obtain an alternative proof.

Proof. Since it is obviously sufficient to consider condition (2) only for $\beta \neq 0$, it is equivalent to the following condition:

(2′) If $\varphi \in F^+$ and $a \in R$ satisfy $\varphi a = \pi$, then there is $\gamma \in F$ with $\gamma a - 1 \in (Y-\alpha)R$.

However, the equivalence (1)$\Leftrightarrow$(2′) follows immediately from Theorem 13 and Lemma 14.

(2)$\Leftrightarrow$(3) Since $\pi(Y-\alpha) = 0$ and $R/(Y-\alpha)R$ is simple, the map $j : R/(Y-\alpha)R \to F^+$, $j(\bar{r}) = \pi r$, is a monomorphism, which is pure iff $\operatorname{Im} j \cap F^+a = (\operatorname{Im} j)a$ for all $a \in R$. As each element of $R/(Y-\alpha)R$ is represented by an element of F, the latter condition is nothing else but (2).

(1)$\Rightarrow$(4) Assume that a_* is injective for some $a \in R$ and let $0 \neq \beta \in F$; we have to show that there is $\gamma \in F$ with $\gamma_* a(Y + \alpha) = \beta$. Since $(a\beta^{-1})_*$ is also injective, it has no root in $\lambda + U$, hence there is $\gamma \in F$ with $\gamma a \beta^{-1} - 1 \in (Y - \alpha)R$. Substituting $Y + \alpha$ for Y yields $\gamma a(Y + \alpha)\beta^{-1} - 1 \in YR$ and $\gamma_* a(Y + \alpha) = \beta$.

(4)$\Rightarrow$(2) Let $\varphi a = \pi\beta$ with $\varphi \in F^+$, $a \in R$ and $\beta \in F$. We have to show the existence of $\gamma \in F$ such that $\gamma a - \beta \in (Y - \alpha)R$ and may assume that $a \neq 0$ and $\beta \neq 0$.

We proceed by induction on $\deg a$. The case $\deg a = 0$ being trivial we assume $\deg a > 0$. If a_* is injective, then ${}_*a(Y + \alpha)$ is surjective, hence there exists $\gamma \in F$ with $\gamma_* a(Y + \alpha) = \beta$ and this means $\gamma a - \beta \in (Y - \alpha)R$. If a_* is not injective, then $a_* \eta = 0$ for some $0 \neq \eta \in F$, thus there exists $b \in R$ with $a\eta = bY$. Hence $\varphi(b_* \xi') = \varphi((bY)_* \xi) = \varphi(a_* \eta \xi) = \pi(\beta \eta \xi)$ for all $\xi \in F$. The special device $\xi = 1$ yields $\pi(\beta\eta) = 0$, i.e. $\beta\eta \in U$, $\beta\eta = (Y - \alpha)_* \zeta = \zeta' - \alpha\zeta$ for some $\zeta \in F$. Thus

$$\varphi(b_* \xi') = \pi((\zeta' - \alpha\zeta)\xi) = \pi((\zeta\xi)' - \alpha\zeta\xi - \zeta\xi') = -\pi(\zeta\xi')$$

for all $\xi \in F$. Now we have to distinguish two cases. If $(Y - \alpha)_*$ is injective, then ${}_*Y$ is surjective, hence $F' = F$ and $\varphi b = -\pi\zeta$. Since $\deg b < \deg a$, the induction hypothesis yields $\gamma \in F$ with $\gamma b + \zeta \in (Y - \alpha)R$. Consequently

$$\begin{aligned} \gamma a \eta - \beta \eta &= \gamma b Y - \beta\eta = (\gamma b + \zeta)Y - \zeta Y - \beta\eta \\ &= (\gamma b + \zeta)Y - (Y - \alpha)\zeta \in (Y - \alpha)R \end{aligned}$$

and $\gamma a - \beta \in (Y - \alpha)R$. If, however, $(Y - \alpha)_*$ is not injective and $0 \neq \nu \in F$ satisfies $\nu' - \alpha\nu = 0$, then $U = \nu F'$ and $\operatorname{Ker} \pi\nu = F'$. Thus there exists $\kappa \in k$ with $\varphi b + \pi\zeta = \pi\nu\kappa$, i.e. $\varphi b = \pi(\nu\kappa - \zeta)$. Now the induction hypothesis yields $\gamma \in F$ with $\gamma b - (\nu\kappa - \zeta) \in (Y - \alpha)R$, and again we can infer $\gamma a \eta - \beta\eta \in (Y - \alpha)R$ and $\gamma a - \beta \in (Y - \alpha)R$. $\square$

A remark concerning the proof of (4)$\Rightarrow$(2) is in order. Unfortunately we are not able to decide whether or not the case that $(Y - \alpha)_*$ is injective can really happen. As we have seen, this alternative implies $F' = F$. To rule out this open problem, and also led by our examples we focus our attention on fields with $F' \neq F$. Now, if $0 \neq \nu \in F$ is a root of $(Y - \alpha)_*$, then $(Y - \alpha)\nu = \nu Y$ and we have the following commutative diagram with exact rows:

$$\begin{array}{lccccccc}
\mathscr{E}: 0 \to & R/(Y-\alpha)R & \xrightarrow{u} & R/Y \frac{1}{\lambda}(Y-\alpha)R & \xrightarrow{v} & F & \to 0 \\
 & {\scriptstyle \iota}\downarrow & & \| & & \| & \\
\mathscr{E}': 0 \longrightarrow & R/YR & \xrightarrow{\quad m \quad} & R/Y \frac{\nu}{\lambda} YR & \xrightarrow{\quad n \quad} & F & \to 0
\end{array}$$

where $t(\bar{r}) = \frac{1}{\nu}\bar{r}$, $m(\bar{r}) = \overline{Y \frac{\nu}{\lambda} r}$ and n is canonical. We intend to formulate anew our existence theorem for an Auslander–Reiten sequence of type $\mathscr{E}'$. First we prove that in the present situation condition (4) possesses a one-sided counterpart.

Lemma 16. *If $F' \neq F$, the following statements are equivalent:*

(1) *If a_* is injective for $a \in R$, then ${}_*a$ is surjective.*

(1′) *If ${}_*a$ is injective, then a_* is surjective.*

(2) *If a_* is injective, then a_* is surjective.*

(2′) *If ${}_*a$ is injective, then ${}_*a$ is surjective.*

Proof. The equivalences (1)⇔(1′) and (2)⇔(2′) are shown by switching from a to $\tilde{a}$.

(1)⇒(2) We prove by induction on $\deg a$, that ${}_*a$ is injective provided that a_* is injective; then (1′) implies that a_* is surjective. Our assertion is obvious if $\deg a = 0$. Now let $\deg a > 0$. We assume that ${}_*a$ is not injective and $0 \neq \mu \in F$ is a root of ${}_*a$; thus there is $b \in R$ such that $\mu a = Yb$. Since a_* is injective, b_* is injective as well, hence ${}_*b$ is injective by induction hypothesis; furthermore, (1) and (1′) imply that ${}_*a$ is surjective and b_* is bijective. Hence the assumption $F' \subsetneq F$ yields the contradiction $F = F_*\mu a = F_*a = F_*Yb = F'_*b \subsetneq F_*b$.

(2)⇒(1) This is proved similarly. □

Corollary 17. *Let $F' \neq F$, $0 \neq \omega \in F$ and the exact sequence*

$$\mathscr{A} : 0 \to F_R \xrightarrow{m} R/Y\tfrac{1}{\omega}YR \xrightarrow{n} F_R \to 0$$

defined by $m(\xi) = \overline{Y\frac{1}{\omega}\xi}$ and the canonical map n. Then the following are equivalent:

(1) *$\mathscr{A}$ is an Auslander–Reiten sequence in m_R.*

(2) *$F = F' \oplus k\omega$, and if a_* is injective for $a \in R$, then a_* is surjective.*

Proof. This follows from Theorems 13 and 15 and Lemma 16. □

We illustrate Corollary 17 by two examples. Let k be a field of characteristic 0, $F = k(T)$ the field of rational functions in one variable T, and $\mathrm{d}/\mathrm{d}T$ the usual derivation on F. Then the field of constants is k and $\dim_k F/F' > 1$, since the family $(1/(\kappa + T))_{\kappa \in k}$ is linearly independent modulo F'. On the other hand, if $F = k((T))$ is the field of formal Laurent series, then $F = F' \oplus k \cdot (1/T)$ and what is more, the second condition of the corollary is valid as a whole (see [9, Section 3]). In the next section we shall represent a class of examples comprising $k((T))$, all of which satisfy the conditions of the latter corollary.

Combining Theorems 10 and 11 and Corollary 17 we obtain the following global result.

Corollary 18. *Let F be a field of characteristic 0 satisfying $F' \neq F$. Then the following statements are equivalent:*

(1) *For each $M \in \mathrm{m}_R$ there exist Auslander–Reiten sequences $0 \to A \to B \to M \to 0$ and $0 \to M \to C \to D \to 0$ in m_R.*

(2) *There is an Auslander–Reiten sequence* $0 \to A' \to B' \to F \to 0$ *in* $\mathfrak{m}_R$.

(3) *There is* $\omega \in F \backslash F'$ *such that* $F = F' \oplus k\omega$ *and each injective differential operator* a_*, $a \in R$, *is surjective.*

Under these conditions the sequence in (2) *is equivalent to the sequence* $\mathcal{A}$ *in Corollary* 17 *and those in* (1) *are equivalent to* $\mathcal{A} \otimes_F M$. *Moreover, each Auslander–Reiten sequence in* $\mathfrak{m}_R$ *is not an Auslander–Reiten sequence in* Mod-R *provided that* $\dim_k F$ *is infinite.* □

Concluding this section we record a remarkable byproduct of Theorem 15.

Corollary 19. *In case the equivalent statements of Theorem* 15 *are valid, then we have*:

(1) *The pure injective hull of* $M \in \mathfrak{m}_R$ *has the form* $j_M : M \to (\mathrm{Hom}_F(M, R/(Y-\alpha)R))^+$.

(2) *The cokernel of* j_M *is an injective module.*

Proof. (1) By assumption there exists a pure embedding $j : R/(Y-\alpha)R \to F^+$. If $M \in \mathfrak{m}_R$ is indecomposable and $N_R = M \otimes_F R/(Y+\alpha)R$, then

$$j \otimes 1_N : R/(Y-\alpha)R \otimes_F N \to F^+ \otimes_F N$$

is pure as well (Lemma 4). It is easy to verify that the source respectively the target of $j \otimes 1_N$ is isomorphic to M_R respectively $(DN)^+ \cong (\mathrm{Hom}_F(M, R/(Y-\alpha)R)^+$. Since M is indecomposable, N and $(DM)^+$ are indecomposable as well, thus we have a pure injective hull $j_M : M \to (\mathrm{Hom}_F(M, R/(Y-\alpha)R))^+$. Now it is obvious that the pure injective hull of an arbitrary module in $\mathfrak{m}_R$ has the same form.

(2) First we show that $Z = \mathrm{Coker}\, j$ is injective. To that end let Ra be a maximal ideal of R. Then

$$j \otimes 1 : R/(Y-\alpha)R \otimes_R R/Ra \to F^+ \otimes_R R/Ra$$

is a monomorphism and

$$F^+ \otimes_R R/Ra \cong \mathrm{Hom}_k(\mathrm{Hom}_R(R/Ra, {}_RF), k)\,.$$

If $R/Ra \cong {}_RF$, then

$$R/(Y-\alpha)R \otimes_R R/Ra \cong F/(Y-\alpha)_*F \cong k$$

and $F^+ \otimes_R R/Ra \cong k$, thus $j \otimes 1$ is an isomorphism. On the other hand, if $R/Ra \not\cong {}_RF$, then $F^+ \otimes_R R/Ra = 0$. In any case $Z \otimes_R R/Ra = 0$, i.e. $Z = Za$. Since each element of R is a product of elements which generate a maximal left ideal, Z

is divisible, thus injective. To show the assertion for M, it is sufficient to prove that $(Z \otimes_F M)_R$ is injective for an injective module Z_R and $M \in \mathfrak{m}_R$. However, this is a consequence of the isomorphism

$$\mathrm{Hom}_R(-, (Z \otimes_F M)_R) \cong \mathrm{Hom}_R(\mathrm{Hom}_F(M, -)_R, Z_R)$$

recorded in Lemma 2. □

If, in addition, we assume $F' \neq F$ in Corollary 19, then we can choose $\alpha = 0$, hence there is a pure injective hull $j_M : M \to (DM)^+$ for each $M \in \mathfrak{m}_R$ and the cokernel of j_M is injective.

3. A class of examples

In this section we present a construction of differential fields $(F, ')$ having the following properties:

(i) $F = F' \oplus k$, where k denotes the field of constants.

(ii) Each injective differential operator a_*, $a \in R$, is surjective.

To that end let k be a field whose additive group contains a linearly ordered subgroup Γ; this forces the characteristic of k to be 0. Let $F = k((\Gamma))$ denote the set of all $\xi = (\xi_\gamma)_{\gamma \in \Gamma} \in k^\Gamma$ having well ordered support $s(\xi) = \{\gamma \in \Gamma \mid \xi_\gamma \neq 0\}$. It is well known that F is a field if the sum of two elements is defined in the obvious way and if the product of $\xi = (\xi_\gamma)_{\gamma \in \Gamma}$ and $\eta = (\eta_\gamma)_{\gamma \in \Gamma}$ in F is given by

$$\xi\eta = \Big(\sum_{\sigma + \tau = \gamma} \xi_\sigma \eta_\tau\Big)_{\gamma \in \Gamma}$$

(see for instance [3, p. 207]; the fact that each non zero element of F is invertible will also be a corollary of our Lemma 21). As it is usual we shall write the elements $\xi = (\xi_\gamma)_{\gamma \in \Gamma}$ of F as power series $\xi = \sum_{\gamma \in \Gamma} \xi_\gamma X^\gamma$. There exists a valuation $v : F \to \Gamma \cup \{+\infty\}$, namely $v(\xi) = \min s(\xi)$ for $\xi \neq 0$ and $v(0) = +\infty$. Furthermore, F carries a derivation, given by

$$\xi' = \sum_{\gamma \in \Gamma} \gamma \xi_\gamma X^\gamma \quad \text{for } \xi = \sum_{\gamma \in \Gamma} \xi_\gamma X^\gamma \in F\,;$$

obviously, the corresponding field of constants is k. As

$$\sum_{\gamma \neq 0} \xi_\gamma X^\gamma = \Big(\sum_{\gamma \neq 0} \frac{1}{\gamma}\, \xi_\gamma X^\gamma\Big)',$$

F' is the set of all $\xi = \sum_\gamma \xi_\gamma X^\gamma \in F$ such that $\xi_0 = 0$, hence $F = F' \oplus k$. Thus, property (i) is settled for $(F, ')$.

It is more difficult to show property (ii). First we need a formula describing the coefficients of $a_*\xi$ as linear functions of the coefficients of ξ. Let

$$0 \neq a = \sum_{k=0}^{n} \alpha_k Y^k \in R$$

$$\text{with } \alpha_k = \sum_{\gamma \in A_k} \alpha_{k\gamma} X^\gamma \in F \text{ and } \xi = \sum_{\gamma \in C} \xi_\gamma X^\gamma \in F\,,$$

where $A_k = s(\alpha_k)$ and $C = s(\xi)$. The kth derivation of ξ is $\xi^{(k)} = \sum_{\gamma \in C} \gamma^k \xi_\gamma X^\gamma$, hence the γth component of $a_*\xi$ equals

$$(a_*\xi)_\gamma = \sum_{\substack{k=0\\ \alpha_k \neq 0}}^{n} \sum_{\substack{\delta \in A_k\\ \gamma-\delta \in C}} \alpha_{k\delta}\xi_{\gamma-\delta}(\gamma-\delta)^k\,.$$

In order to simplify this expression, let $\nu = \min\{v(\alpha_k) \mid 0 \le k \le n\}$ and $I = \{k \mid 0 \le k \le n,\ \alpha_k \neq 0 \text{ and } \nu = v(\alpha_k)\}$. Then

$$\begin{aligned}(a_*\xi)_\gamma &= \sum_{k\in I} \alpha_{k\nu}(\gamma-\nu)^k \xi_{\gamma-\nu} + p_\gamma((\xi_{\gamma-\rho})_{\gamma-\rho \in C, \rho > \nu})\\ &= q(\gamma-\nu)\xi_{\gamma-\nu} + p_\gamma((\xi_{\gamma-\rho})_{\gamma-\rho\in C,\rho>\nu})\,,\end{aligned}$$

where $q(T) = \sum_{k\in I} \alpha_{k\nu}T^k \in k[T]$ is a non zero polynomial of degree $\le n$ and the second summand a linear function of $(\xi_{\gamma-\rho})_{\gamma-\rho\in C,\rho>\nu}$.

Lemma 20. *If $0 \neq \xi \in F$ satisfies $q(v(\xi)) \neq 0$, then $v(a_*\xi) = v(\xi) + \nu$.*

Proof. We have $(a_*\xi)_{v(\xi)+\nu} = q(v(\xi))\xi_{v(\xi)} \neq 0$ and $(a_*\xi)_\gamma = 0$ for $\gamma < v(\xi) + \nu$. □

Lemma 21. *Let $\tilde\gamma \in \Gamma$ such that $q(\gamma) \neq 0$ for all $\gamma \in \Gamma$, $\gamma > \tilde\gamma$. Then the equation $a_*\xi = \eta$ possesses a solution $\xi \in F$ with $v(\xi) > \tilde\gamma$ for all $\eta \in F$ with $v(\eta) > \tilde\gamma + \nu$.*

Since any constant $0 \neq a \in F$ satisfies the assumption of this lemma, it yields in particular that F is a field. In fact, to prove it we shall use a modification of the argument that F is a field in Hausdorff's classic *Grundzüge der Mengenlehre* [3, p. 207].

Proof. By considering $X^{-\nu}a$ and $X^{-\nu}\eta$ instead of a and η, we can assume that $\nu = 0$. So let $\nu = 0$ and $0 \neq \eta \in F$ with $v(\eta) > \tilde\gamma$. We assume that $a_*\xi \neq \eta$ for all $\xi \in F$ with $v(\xi) > \tilde\gamma$ and denote by S the set of all $v(a_*\xi - \eta)$, where $v(\xi) > \tilde\gamma$. We have to distinguish two cases.

(a) S possesses a largest element $\delta = v(a_*\sigma - \eta)$. Since $v(\sigma) > \tilde\gamma$, $q(v(\sigma)) \neq 0$, thus $v(a_*\sigma) = v(\sigma)$ and $\delta = v(a_*\sigma - \eta) \ge \min\{v(a_*\sigma), v(\eta)\} > \tilde\gamma$. Let $\tilde\sigma =$

$\sigma - \kappa_\delta q(\delta)^{-1}X^\delta$, where $\kappa_\delta \neq 0$ is the lowest coefficient of $a_*\sigma - \eta$. Then $v(\tilde{\sigma}) > \tilde{\gamma}$, hence $v(a_*\tilde{\sigma} - \eta) \in S$. Because $q(\delta)$ is the lowest coefficient of a_*X^δ we obtain the contradiction $v(a_*\tilde{\sigma} - \eta) > \delta$.

(b) S possesses no largest element. In this case, let T be the set of all $\gamma \in \Gamma$, $\gamma > \tilde{\gamma}$, for which there exists $\gamma^* \in S$ with $\gamma < \gamma^*$. We need the following fact: If $\tau \in T$ and $\xi_1, \xi_2 \in F$ are different elements satisfying $v(\xi_i) > \tilde{\gamma}$ and $\tau < v(a_*\xi_i - \eta)$, $i = 1,2$, then $v(\xi_1 - \xi_2) = v((a_*\xi_1 - \eta) - (a_*\xi_2 - \eta)) > \tau$, thus $\xi_{1\tau} = \xi_{2\tau}$. Therefore, we can define $\lambda \in F$ by $\lambda_\tau = \xi_\tau$ for $\tau \in T$, where $\xi \in F$ is some element with $v(\xi) > \tilde{\gamma}$ and $\tau < v(a_*\xi - \eta)$, and $\lambda_\gamma = 0$ for $\gamma \in \Gamma \setminus T$. We skip the easy proof that $s(\lambda)$ is well ordered. Obviously $v(\lambda) > \tilde{\gamma}$, thus $\delta = v(a_*\lambda - \eta) \in S$. Because there is no largest element in S, there exists $\tilde{\lambda} \in F$ with $v(\tilde{\lambda}) > \tilde{\gamma}$ and $\delta < v(a_*\tilde{\lambda} - \eta)$. By construction of λ we have $\tilde{\lambda}_\delta = \lambda_\delta$; on the other hand, $v(\tilde{\lambda} - \lambda) = v((a_*\tilde{\lambda} - \eta) - (a_*\lambda - \eta)) = \delta$, thus we have the contradiction $\tilde{\lambda}_\delta \neq \lambda_\delta$. □

Theorem 22. *Let $0 \neq a \in R$ and $q \in k[T]$ be the previously defined polynomial. Then the following statements are equivalent:*

(1) *a_* is injective.*

(2) *The equation $a_*\xi = \eta$ is uniquely solvable in F for all $\eta \in F$.*

(3) *q has no root in Γ.*

Proof. As in the preceding proof we may assume that $\nu = 0$.

(1)⇒(3) Suppose that q has a root in Γ. Then q has a largest root, say $\rho \in \Gamma$, hence $v(a_*X^\rho) > \rho$. Since $q(\gamma) \neq 0$ for all $\gamma > \rho$, there is $\xi \in F$ with $v(\xi) > \rho$ and $a_*\xi = a_*X^\rho$ (Lemma 21). As $\xi \neq X^\rho$, the latter equation contradicts (1).

(3)⇒(1) In case a_* is not injective and $0 \neq \xi \in F$ is a root of a_*, $\gamma = v(\xi)$, then $0 = (a_*\xi)_\gamma = q(\gamma)\xi_\gamma$, hence $q(\gamma) = 0$.

The implication (3)⇒(2) is a consequence of Lemma 21 and the implication (2)⇒(1) is obvious. □

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